Lecture 3

Wednesday, 24 August 2022 10:33 AM

Finding equilibria in zero-sum games:

Let (R, C) be a zero-sum game. Thus, $C = -R^T$.
Consider the column player's perspective. Suppose it plays y. Then the row player's utilities for it's strategies are Ry (this is a column vector).
If the row player chooses best-response to y, it gets $\max_i (Ry)_i$, and hence the column player gets $-\max_i (Ry)_i$.

Since at equilibrium both players best-respond to each other (by defn.), the column player "should" choose y to maximize it's utility when row player best-responds, i.e., choose y to maximize $-\max_i (Ry)_i$

$$\equiv \text{maximize}_y \min_i (-Ry)_i$$

Note that we are not saying that such a y is an equilibrium strategy, in particular why y is a best-response to the row-player's strategy (it may not be!)

But we can find such a y by an LP:

$$P_C: \quad \begin{aligned} \max\ & z \\ \text{s.t. } \forall i,\ (-Ry)_i & \geq z \\ \sum_j y_j & = 1 \\ y & \geq 0 \end{aligned}$$

Similarly, for the row player, a good strategy would be to choose x which optimizes:

$$P_R: \quad \begin{aligned} \max\ & w \\ \text{s.t. } \forall j,\ (-Cx)_j & \geq w \\ \sum_i x_i & = 1 \\ x & \geq 0 \end{aligned}$$

(rewriting P_R):

$$\begin{aligned} \max\ & w \\ y\times \quad Cx + w & \leq 0 \\ z'\times \quad \sum_i x_i & = 1 \\ x & \geq 0 \end{aligned}$$

Let us write the dual of P_R. This is:

$$\begin{aligned} \min\ & z' \\ \forall i,\ (C^T y)_i & \geq -z' \\ \sum_j y_j & = 1 \\ y & \geq 0 \end{aligned} \quad \equiv \quad D_R: \begin{aligned} \min\ & -z \\ \forall i,\ (-Ry)_i & \geq z \\ \sum_j y_j & = 1 \\ y & \geq 0 \end{aligned}$$

Note that D_R is nearly the same as P_C, except that the objective value gets negated. I.e., (y^*, z^*) is optimal for D_R iff (y^*, z^*) is optimal for P_C.

Let (x^*, w^*) be optimal for P_R, & (y^*, z^*) be optimal for P_C. Then by <u>strong</u> duality, $-z^* = w^*$.

Theorem: (x^*, y^*) is a NE

<u>Proof</u>: We need to show that for the row player, x^* is a best-response to y^*, i.e., $\forall x,\ x^T R y^* \leq x^{*T} R y^*$.

Consider y^*. We know that if column player plays y^*, and if row-player best-responds, column player gets z^* (negation of D_R). Thus, row-player gets $-z^*$. Thus for any response to y^*, row-player gets at most $-z^*$.

$$\forall x \quad x^T R y^* \leq -z^* = w^*$$

Now consider x^*, Similar to above, for any strategy y, row player gets at least w^*.

$$\forall y \quad x^{*T} R y^* \geq w^*.$$

Thus, $x^{*T} R y^* \geq x^T R y^* \ \forall x$, and hence x^* is a best response to y^*.
Similarly we can show that y^* is a best-response to x^*, and hence (x^*, y^*) is a NE ∎

Note:
1. The proof holds for <u>any</u> optimal soln. x^* to P_R, and <u>any</u> optimal soln. y^* to P_C.
2. For any such x^*, y^*, the row-player's utility at equilibrium is w^*. Hence, there are multiple equilibria, but the row-player's payoff (and hence, the column player's payoff) is exactly the same.
   The value w^* is called the value of the zero-sum game
3. At equilibrium, each player is playing a min-max strategy, or a risk-averse strategy.
   In general games, a min-max strategy does not give an equilibrium.

Theorem: Let (x^*, y^*) be a NE of a zero-sum game, & w^*, z^* be payoffs of the two players.
Then (x^*, w^*) is an optimal soln. for P_R, and (y^*, z^*) is an optimal soln for P_C.

(Q) Prove yourself.

Computing Equilibria in General Bimatrix Games.

Let (R, C) be a general bimatrix game, $R, C \in \mathbb{R}^{m \times n}$
Thus row player has m pure strategies, column player has n pure strategies, and $x \in \Delta_m$, $y \in \Delta_n$ are mixed strategies.
We use e_i to denote the column vector $[0 \dots 0\ 1\ 0 \dots 0]^T$ (1 at position i).

We will now give an exponential-time algorithm for computing equilibria in bimatrix games.
Recall: given $x \in \Delta_m$, $\text{Supp}(x) = \{i : x_i > 0\}$
Similarly for $y \in \Delta_n$.

Fix $S_R \subseteq [m]$, $S_C \subseteq [n]$ as subsets of pure strategies for the players. Consider the following LP:

$P(S_R, S_C)$:

$$\begin{aligned} \max\ & 0 \\ \text{s.t. } & x \in \Delta_m \\ & y \in \Delta_n \\ \forall i \notin S_R,\ & x_i = 0 \\ \forall j \notin S_C,\ & y_j = 0 \\ \forall i \in S_R, i' \in [m],\ & (Ry)_i \geq (Ry)_{i'} \\ \forall j \in S_C, j' \in [n],\ & (Cx)_j \geq (Cx)_{j'} \end{aligned}$$

(the two constraints on x_i, y_j are marked A; the last two are marked B)

So: (A) x is supported on S_R, y on S_C
(B) S_R is a subset of best-responses to y
S_C is a subset of best-responses to x

Theorem: (i) If (x^*, y^*) is a feasible soln to $P(S_R, S_C)$, then (x^*, y^*) is a NE of the game.
(ii) If (x^*, y^*) is a NE, let $S_R^* = \text{supp}(x^*)$, $S_C^* = \text{supp}(y^*)$. (x^*, y^*) is a feasible soln. to $P(S_R^*, S_C^*)$.

<u>Proof of (ii)</u>: easy.

Note:
1. Every LP w/ rational coefficients has a rational soln. Hence, if utilities R, C are rational, the game has a rational equilibrium.
2. Let (x_1^*, y^*), (x_2^*, y^*) be two equilibria of (R, C). Then $\forall\ 0 \leq \lambda \leq 1$, $(\lambda x_1^* + (1-\lambda) x_2^*, y^*)$ is also an equilibrium.

(Q.) Prove (ii) yourself.

Proof of theorem (i):

<u>Claim 1</u>: (x^*, y^*) is a NE iff

$$x^{*T} R y^* \geq e_i^T R y^* \quad \forall\ i \in [m]$$
$$y^{*T} C x^* \geq e_j^T C x^* \quad \forall j \in [n]$$

<u>Proof</u>: Easy.

<u>Corollary</u>: (x^*, y^*) is a NE iff

$$\text{supp}(x^*) \subseteq \arg\max_{i \in [m]} (Ry^*)_i$$
$$\& \ \text{supp}(y^*) \subseteq \arg\max_{j \in [n]} (Cx^*)_j$$

Proof of Theorem: By the constraints:

$$\text{supp}(x^*) \overset{A}{\subseteq} S_R \overset{B}{\subseteq} \arg\max_{i \in [m]} (Ry^*)_i$$
$$\text{supp}(y^*) \overset{A}{\subseteq} S_C \overset{B}{\subseteq} \arg\max_{j \in [n]} (Cx^*)_j$$

Hence, (x^*, y^*) is a NE by the corollary.

By Nash's Theorem, we know $\exists$ a NE, hence $\exists\ S_R, S_C$ for which $P(S_R, S_C)$ is feasible.
Algorithm enumerates over all possible $S_R \subseteq [m]$, $S_C \subseteq [n]$, solves $P(S_R, S_C)$ to check if feasible, takes time

$$\text{poly}(m, n, |R|, |C|)\ 2^{m+n}$$